TIME EVOLUTION IN THE EXTERNAL FIELD: THE UNITARITY PARADOX

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Abstract

One of the axioms of quantum field theory is the property of unitarity of the evolution operator. However, if one considers the quantum electrodynamics in the external field in the leading order of perturbation theory, one will find that the evolution transformation is a non-unitary canonical transformation of creation and annihilation operators. This observation was one of the arguments for the hypothesis that one should choose different representations of the canonical commutation relations at different moments of time in the exact quantum field theory. In this paper the contradiction is analyzed for the case of a simple quantum mechanical model being an analog of the leading order of the large-N field theory. On the one hand, this model is renormalized with the help of the constructive field theory methods; the Hilbert space and unitary evolution operator are constructed. On the other hand, the leading order of the evolution transformation in the strong external field is shown to be non-unitary. Thus, unitarity of evolution in the exact theory is not in contradiction with non-unitarity of the approximate theory.

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1 Introduction

One of the main postulates of the axiomatic relativistic quantum field theory in the Wightman approach [1, 2] (for a review see [3]) is the following. The Poincare transformations should be unitary operators in the Hilbert state space. For example, this postulate should be correct for the operator of time evolution. However, the check of this axiom for realistic models of QFT is difficult, since the models are usually constructed in the perturbative approach only.

On the other hand, one can investigate the (3+1)-dimensional spinor QED in the strong external classical electromagnetic field [4, 5, 6, 7]. It happens that even in the leading order of perturbation theory the creation and annihilation operators at different moments of time are related with the help of *non-unitary* Bogoliubov canonical transformation. This means that for constructing QFT in the external field, it is necessary to use different representations of the canonical commutation relations (CCR) at different moments of time.

This observation implies that one can expect that in the non-perturbative QFT one should also consider different representations of CCR at different moments of time [9], while the time translation (evolution) is *not* an unitary operator in the Hilbert space but transformation connecting different representations of CCR. This suggestion is in contradiction with the Wightman axiomatic approach. However, it is in agreement with the more general algebraic approach developed by Haag and Kastler [10] (for recent reviews of the algebraic approach see [11, 12]).

It is shown in this paper that non-unitarity of the evolution operator in the external field in the leading order of perturbation theory does *not* contradict to the unitarity axiom of the "exact" theory. The simple exactly solvable model is considered in this paper. The Hilbert state space and *unitary* evolution operator are constructed with the help of the Bogoliubov S-matrix approach [13, 14]. This model can be also considered in the strong external field in the leading order of perturbation theory: the corresponding states are constructed. The evolution transformation is a *non-unitary* canonical transformation.

This model corresponds to the (0+1)-dimensional "field" $Q(t)$ interacting with infinite number of "fields" $Q_k(t)$. The action of the model is

$$S = \int dt [L_Q + \sum_{k=1}^{\infty} \left(\frac{\dot{q}_k^2}{2} - \frac{\Omega_k^2 q_k^2}{2} \right) - g \sum_{k=1}^{\infty} \mu_k q_k Q], \tag{1}$$

where

$$L_Q = \sum_{s=0}^{l} (-1)^{s+1} z_s (Q^{(s)})^2, \qquad Q^{(s)} = \frac{d^s Q}{dt^s},$$

while μ_k are some coefficients. As $k \to \infty$, the set of numbers Ω_k tends to infinity. If $\sum_{k=1}^{\infty} \frac{\mu_k^2}{\Omega_k^2} = \infty$, the problem of divergences arises. However, if $\sum_{k=1}^{\infty} \frac{\mu_k^2}{\Omega_k^m} = \infty$ for some m, the divergences can be renormalized.

Note that the model (1) is a quantum-mechanical analog of the large-N theory $\Phi\varphi^a\varphi^a$. Large-N conception is very useful in QFT: one can perform resummation of Feynman graphs, making use of the diagram and functional-integral techniques [15, 16, 17]. The problem of external field (the back reaction) can be also investigated in the large-N theory [18, 19].

Consider the theory of N fields φ^a interacting with the field Φ in the $(d+1)$-dimensional space-time. The Lagrangian of the theory is

$$\mathcal{L} = \sum_{a=1}^N : \left(\tfrac{1}{2}\partial_\mu\varphi^a\partial_\mu\varphi^a - \tfrac{\mu^2}{2}\varphi^a\varphi^a \right) : + \tfrac{z}{2}\partial_\mu\Phi\partial_\mu\Phi - \tfrac{M^2}{2}\Phi^2$$
$$- \tfrac{g}{\sqrt{N}} : \left(\sum_{a=1}^N \varphi^a\varphi^a \right) : \Phi$$

Analogously to [20] (see also [21, 22, 23, 24]), introduce the "collective fields" being the operators of creation and annihilation of pairs of particles

$$A^{\pm}_{\mathbf{kp}} = \frac{1}{\sqrt{2N}} \sum_{a=1}^{N} b^{\pm a}_{\mathbf{k}} b^{\pm a}_{\mathbf{p}},$$

where $b^{\pm a}_{\mathbf{k}}$ is a creation-annihilation operator of the particle with momentum $\mathbf{k}$, which corresponds to the field φ^a.

We will consider the states of the N-field theory which depend on the large parameter N as follows

$$\sum_n \int d\mathbf{k}_1 d\mathbf{p}_1 ... d\mathbf{k}_n d\mathbf{k}_n A^+_{\mathbf{k}_1\mathbf{p}_1} ... A^+_{\mathbf{k}_n\mathbf{p}_n} \chi^n_{\mathbf{k}_1\mathbf{p}_1...\mathbf{k}_n\mathbf{p}_n} \Psi, \tag{2}$$

with regular as $N \to \infty$ coefficient functions χ^n and such vector Ψ that does not contain the particles corresponding to the fields φ^a.

Note that operators of the form

$$\int d\mathbf{k}d\mathbf{p} \frac{1}{\sqrt{N}} \sum_{a=1}^{N} b^{+a}_{\mathbf{k}} b^{-a}_{\mathbf{p}} \varphi_{\mathbf{kp}}$$

multiply the norm of the state (2) by the quantity $O(N^{-1/2})$. Therefore, they can be neglected as $N \to \infty$. In this approximation

$$[A^-_{\mathbf{k}_1\mathbf{p}_1}; A^+_{\mathbf{k}_2\mathbf{p}_2}] \simeq \frac{1}{2}(\delta_{\mathbf{k}_1\mathbf{k}_2}\delta_{\mathbf{p}_1\mathbf{p}_2} + \delta_{\mathbf{k}_1\mathbf{p}_2}\delta_{\mathbf{k}_2\mathbf{p}_1}).$$

Consider the free Hamiltonian $H_0 = \int d\mathbf{k}\omega_{\mathbf{k}} \sum_{a=1}^{N} b^{+a}_{\mathbf{k}} b^{-a}_{\mathbf{k}}$, where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + \mu^2}$. If we consider the states of the form (2) only, it coincides with the operator

$$\int d\mathbf{k}d\mathbf{p} A^+_{\mathbf{kp}}(\omega_{\mathbf{k}} + \omega_{\mathbf{p}}) A^-_{\mathbf{kp}},$$

The operator $\frac{1}{\sqrt{N}} \sum_{a=1}^{N} \varphi^a(\mathbf{x})\varphi^a(\mathbf{x})$ is approximately equal to

$$\frac{\sqrt{2}}{(2\pi)^d} \int \frac{d\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \frac{d\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}} (A^+_{\mathbf{kp}} e^{-i(\mathbf{k}+\mathbf{p})\mathbf{x}} + A^-_{\mathbf{kp}} e^{i(\mathbf{k}+\mathbf{p})\mathbf{x}}).$$

The leading order for the Hamiltonian in $1/N$ is analogous to eq.(1):

$$\begin{aligned} H = \int d\mathbf{k}d\mathbf{p} A^+_{\mathbf{kp}}(\omega_{\mathbf{k}} + \omega_{\mathbf{p}}) A^-_{\mathbf{kp}} + \int d\mathbf{x} \left(\tfrac{1}{2z}\Pi^2(\mathbf{x}) + \tfrac{z}{2}(\nabla\Phi)^2(\mathbf{x}) + \tfrac{M^2}{2}\Phi^2 \right) \\ + \tfrac{\sqrt{2}g}{(2\pi)^d} \int d\mathbf{x} \left[\int \tfrac{d\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \tfrac{d\mathbf{p}}{\sqrt{2\omega_{\mathbf{p}}}} (A^+_{\mathbf{kp}} e^{-i(\mathbf{k}+\mathbf{p})\mathbf{x}} + A^-_{\mathbf{kp}} e^{i(\mathbf{k}+\mathbf{p})\mathbf{x}}) \right] \Phi(\mathbf{x}). \end{aligned} \tag{3}$$

The index k is substituted by $(\mathbf{k}, \mathbf{p})$, the sums are substituted by integrals. Eq.(3) can be also obtained from the third-quantized approach [25, 26].

Investigation of the models of the type (1) allows us to understand the difficulties of the quantum field theory in the external field.

This paper is organized as follows. In section 2 we construct the Hilbert state space and unitary evolution operator for the model (1) which occurs to be renormalizable. Section 3 deals with constructing special states of the model (1) which correspond to the strong classical external field. Time evolution in the obtained quantum theory in the external field is shown to be a non-unitary canonical transformation. Section 4 is devoted to the analysis of the paradox.

2 Construction of the model: evolution as an unitary transformation

In this section the quantized model (1) is constructed. We show that the evolution operator is a well-defined unitary transformation in the Hilbert state space.

2.1 The Bogoliubov S-matrix and unitarity of evolution

2.1.1 Conditions on the Hamiltonian and on the state

Models of the constructive field theory are usually formulated as follows [28, 29]. Instead of the Schrodinger equation obtained by the formal quantization procedure,

$$i\dot{\Psi}_t = [\hat{H}_0 + g\hat{H}_1]\Psi_t, \tag{4}$$

where $\hat{H}_0$ is a free Hamiltonian, $\hat{H}_1$ is an interaction containing UV divergences, one considers the regularized equation. The Hamiltonian $\hat{H}_1$ is substituted by the regular operator $\hat{H}_1^\Lambda$ depending on the cutoff parameter Λ. At finite values of Λ this operator does not contain UV-divergences. As $\Lambda \to \infty$, the regularized expression for the Hamiltonian formally tends to $\hat{H}_1$. Usually, it is necessary to add the counterterms $\hat{H}_{ct}^\Lambda(g)$ to the Hamiltonian. The evolution equation reads,

$$i\dot{\Psi}_t^\Lambda = \hat{H}^\Lambda \Psi_t^\Lambda = [\hat{H}_0 + g\hat{H}_1^\Lambda + \hat{H}_{ct}^\Lambda(g)]\Psi_t^\Lambda. \tag{5}$$

In the S-matrix approach one imposes the conditions on the dependence of the counterterms on the cutoff parameter in order to obtain the finite S-matrix. Within the perturbation framework, it is possible: the well-known Bogoliubov-Parasiuk theorem on renormalizability of QFT if proved. Contrary to the S-matrix approach, in order to eliminate divergences from the equations of motion, it is *not* sufficient to impose conditions on the counterterms. Even in the tree Feynman graphs the Stueckelberg divergences arise [30] when one investigates the processes at finite time intervals (like emission of the virtual photon by a single electron [30]). To eliminate the Stueckelberg divergences, one should also impose the conditions on the dependence on Λ of the initial condition for eq.(5). The problem of elimination of the Stueckelberg divergences for the leading order of semiclassical expansion was investigated in [31, 32, 33].

In the constructive field theory [28, 34] one usually chooses such t-independent unitary transformation T_Λ (singular as $\Lambda \to \infty$) that the following requirement is satisfied. Suppose that the initial condition for eq.(5) depends on Λ as $\Psi_0^\Lambda = T_\Lambda \Phi_0^\Lambda$, where the vector Φ_0^Λ has a strong limit as $\Lambda \to \infty$. Then the solution to the Cauchy problem for eq.(5) should have an analogous form:

$$\Psi_t^\Lambda = T_\Lambda \Phi_t^\Lambda. \tag{6}$$

with regular as $\Lambda \to \infty$ vector Φ_t^Λ, $\Phi_t^\Lambda \to_{\Lambda\to\infty} \Phi_t$. The vector Φ_t can be viewed as a "renormalized" state. The operator transforming the state Φ_0 to the state Φ_t is regular as $\Lambda \to \infty$,

$$U_t = s - \lim_{\Lambda\to\infty} U_\Lambda^t = \lim_{\Lambda\to\infty} (T_\Lambda)^{-1} \exp[-i\hat{H}_\Lambda t] T_\Lambda \tag{7}$$

It can be viewed as a renormalized evolution operator in the model (4).

Note that the evolution operator $\exp[-i\hat{H}_\Lambda t]$ may be singular as $\Lambda \to \infty$ while the renormalized operator (7) may be regular.

Consider the arbitrary observable O corresponding to the operator O_Λ in the regularized theory. In the representation (6) it can be written as:

$$T_\Lambda^{-1} \hat{O}_\Lambda T_\Lambda. \tag{8}$$

If eq.(8) possesses the limit $\Lambda \to \infty$, one can talk about the *time-independent* representation of the observable O in the "renormalized" state space. In particular, this is a way to construct a time-independent non-Fock representation of the canonical commutation relation.

2.1.2 The Bogoliubov approach

To construct the operator T_Λ, let us use the axiomatic Bogoliubov approach [13] (see also [14]) based on the conception of switching on the interaction. In this approach one considers the analog of the model (4) with the *time-dependent* coefficient of interaction $g_t = g(t)$ instead of the case of the constant interaction. This generalization of the model seems to be a complication. However, if one considers the case of the smooth function g which is non-zero only on the finite time interval, the S-matrix will be regular as $\Lambda \to \infty$, contrary to the evolution operator which can be viewed as S-matrix corresponding to the discontinuous function g being constant at $t \in (t_1, t_2)$ and zero at $t \notin (t_1, t_2)$. The S-matrix viewed as a functional on the smooth function g vanishing at sufficiently large $|t|$ is the main notion of the Bogoliubov axiomatic approach. One can determine the renormalized evolution operator in terms of S-matrix [14].

After regularization and renormalization the Bogoliubov S-matrix takes the form

$$S_\Lambda[g] = Texp(-i \int_{-\infty}^{\infty} e^{i\hat{H}_0\tau}(g(\tau)\hat{H}_1^\Lambda + \hat{H}_{ct}^\Lambda[\tau, g(\cdot)])e^{-i\hat{H}_0\tau} d\tau). \qquad (9)$$

It transforms the initial condition for the equation

$$i\frac{d\tilde{\Phi}_\Lambda^t}{dt} = e^{i\hat{H}_0 t}(g(t)\hat{H}_1^\Lambda + \hat{H}_{ct}^\Lambda[t, g(\cdot)])e^{-i\hat{H}_0 t}\tilde{\Phi}_\Lambda^t \qquad (10)$$

as $t = -\infty$ to the solution of this equation at $t = +\infty$, $S_\Lambda(g)\tilde{\Phi}^{-\infty} = \tilde{\Phi}^{+\infty}$. Note the function $g(t)$ is zero at $|t| > T$.

The counterterms $\hat{H}_{ct}^\Lambda[t, g(\cdot)])$ depending on the function g and its derivatives at time moment t are chosen in order to make the S-matrix regular. More precisely, for smooth functions $g(t)$ the S-matrix should have a strong limit as $\Lambda \to \infty$.

In the interaction representation for finite values of Λ the evolution operator coincides with the Bogoliubov S-matrix (9) if $g(\tau) = g$ at $\tau \in [t_1, t_2]$ and $g(\tau) = 0$ at $\tau \notin [t_1, t_2]$. Namely, the substitution $\Psi_\Lambda^t = e^{-i\hat{H}_0 t}\tilde{\Phi}_\Lambda^t$ transforms eq.(5) to the form (9), so that

$$e^{-i\hat{H}_\Lambda(t_2 - t_1)} = e^{-i\hat{H}_0 t_2} S_\Lambda[gI_{t_1 t_2}(\cdot)] e^{i\hat{H}_0 t_1},$$

where $I_{t_1 t_2}(t) = 1$ as $t \in (t_1, t_2)$ $I_{t_1 t_2}(t) = 0$ as $t \notin (t_1, t_2)$. However, because of the Stueckelberg divergences [30, 14] the strong limit of the S-matrix as $\Lambda \to \infty$ for the case of a discontinuous function g, in general, does not exist.

Consider the function $\xi_-(\tau)$ which switches on from 0 to 1 at $-T_1 < \tau < -T_2$, $-T_2 < 0$, is equal to 1 at $-T_2 < \tau < 0$ and 0 at $\tau > 0$ and $\tau < -T_1$ (see fig.1).

Choose the unitary operator T_Λ to be the following:

$$T_\Lambda = S_\Lambda[g\xi_-(\cdot)]. \qquad (11)$$

Consider also the function $\xi_+(\tau) = \xi_-(-\tau)$ and operator v_t of shifting the argument τ: $v_t g(\tau) = g(\tau + t)$. The operator U_Λ^t entering to eq.(7) takes the form:

$$U_\Lambda^{t_2 - t_1} = S_\Lambda^+[g\xi_-(\cdot)] e^{-i\hat{H}_0 t_2} S_\Lambda[gI_{t_1 t_2}(\cdot)] e^{i\hat{H}_0 t_1} S_\Lambda[g\xi_-(\cdot)].$$

The property $S_\Lambda[v_t g] = e^{i\hat{H}_0 t} S_\Lambda[g] e^{-i\hat{H}_0 t}$ and unitarity of the operator $S_\Lambda[g\xi_+(\cdot)]$ imply that

$$\begin{gathered} U_\Lambda^{t_2-t_1} = e^{-i\hat{H}_0 t_2} S_\Lambda^+[g v_{t_2}\xi_-(\cdot)] S_\Lambda^+[g v_{t_2}\xi_+(\cdot)] \\ \times S_\Lambda[g v_{t_2}\xi_+(\cdot)] S_\Lambda[g I_{t_1 t_2}(\cdot)] S_\Lambda[g v_{t_1}\xi_-(\cdot)] e^{i\hat{H}_0 t_1}. \end{gathered} \tag{12}$$

Denote as $\xi_{t_1 t_2}$ the smooth function of the form

$$\xi_{t_1 t_2} = v_{t_1}\xi_- + I_{t_1 t_2} + v_{t_2}\xi_+, \qquad t_1 \le t_2.$$

The operator (12) can be presenter as

$$U_\Lambda^{t_2-t_1} = e^{-i\hat{H}_0 t_2} S_\Lambda^+[g\xi_{t_2 t_2}(\cdot)] S_\Lambda[g\xi_{t_1 t_2}(\cdot)] e^{i\hat{H}_0 t_1}.$$

Thus, the operator (12) is expressed via the values of the Bogoliubov S-matrix functional on the smooth functions g which vanish at $|t| > T$ for some T.

In the next subsection we show that the operators $S_\Lambda[g(\cdot)]$ and $S_\Lambda^+[g(\cdot)]$ have strong limits as $\Lambda \to \infty$.

This will imply that the *renormalized* evolution operators (7) exist.

Note that two operators T_Λ corresponding to different functions of switching the interaction $\xi_-^{(1)}$ and $\xi_-^{(2)}$ lead to equivalent representations of the observables since the unitary operator $S_\Lambda^+[g\xi_-^{(1)}] S_\Lambda[g\xi_-^{(2)}] = S_\Lambda^+[g(\xi_-^{(1)} + \xi_+^{(1)})] S_\Lambda[g(\xi_-^{(2)} + \xi_+^{(1)})]$ has a (strong) limit as $\Lambda \to \infty$.

2.2 Construction of the Bogoliubov S-matrix

2.2.1 Regularization and canonical quantization

Consider the canonical quantization of the model (1). Since the Lagrangian contains higher derivatives, the classical Hamiltonian of the model depends [35] on the coordinates $V_0 = Q$, $V_1 = \dot{Q}$, ..., $V_{l-1} = Q^{(l-1)}$ and canonically conjugated momenta P_0, P_1, ..., P_{l-1}, as well as on the coordinates and momenta q_k and p_k. The classical Hamiltonian function has the form:

$$H = H_q + H_Q + g\sum_{k=1}^{\infty} \mu_k q_k Q, \tag{13}$$

where

$$\begin{gathered} H_q = \sum_{k=1}^{\infty}\left(\frac{p_k^2}{2} + \frac{\Omega_k^2 q_k^2}{2}\right), \\ H_Q = \frac{(-1)^{l+1}P_{l-1}^2}{2z_l} + \sum_{s=0}^{l-2} P_s V_{s+1} + \sum_{s=0}^{l-1} \frac{(-1)^s}{2} z_s V_s^2. \end{gathered}$$

Under the canonical quantization procedure, the coordinates and momenta are associated with the operators $\hat{V}_0, ..., \hat{V}_{l-1}$, $\hat{P}_0, ..., \hat{P}_{l-1}$, $\hat{q}_k$, $\hat{p}_k$ obeying the canonical commutation relations (CCR):

$$[\hat{V}_m, \hat{P}_k] = i\delta_{mk}, \qquad [\hat{q}_k, \hat{p}_l] = i\delta_{kl}. \tag{14}$$

Other commutators vanish.

In order to avoid the divergences at the intermediate stages of the analysis of the model, introduce the regularization. The quantities μ_k are substituted by

$$\mu_k^\Lambda = \mu_k; \quad k < \Lambda, \qquad \mu_k^\Lambda = 0, \quad k \ge \Lambda.$$

where Λ is a cutoff parameter. We will show that the Bogoliubov S-matrix is regular as $\Lambda \to \infty$ if the counterterm

$$H_{ct}^\Lambda = \sum_{s=0}^{l-1} \frac{(-1)^s}{2} \delta z_s V_s^2.$$

is added to the Hamiltonian (13).

In the cutoffed theory one can use the Fock representation of CCR (14). The operators $\hat{p}_k$ and $\hat{q}_k$ are presented via the creation and annihilation operators $\hat{a}_k^{\pm}$ in the Fock space:

$$\hat{q}_k = \frac{\hat{a}_k^+ + \hat{a}_k^-}{\sqrt{2\Omega_k}}; \hat{p}_k = i\sqrt{\frac{\Omega_k}{2}}(\hat{a}_k^+ - \hat{a}_k^-), \tag{15}$$

They obey the following relations:

$$[\hat{a}_k^-, \hat{a}_m^+] = \delta_{km}, \quad [\hat{a}_k^{\pm}, \hat{a}_l^{\pm}] = 0.$$

Remind that the Fock space $\mathcal{F}$ is a space of sets

$$\Psi = (\Psi_0, \Psi_1(k_1), \Psi_2(k_1, k_2), ...)$$

of symmetric with respect to $k_1, ..., k_n$ functions $\Psi_n(k_1, ..., k_n)$, $k_1, ..., k_n = 1, 2, 3, ...$, such that the series

$$\sum_{n=0}^{\infty} \sum_{k_1...k_n=1}^{\infty} |\Psi_n(k_1, ..., k_n)|^2 < \infty \tag{16}$$

converges. The operators $\hat{a}_k^{\pm}$ act in the Fock space as

$$(\hat{a}_k^+ \Psi)_n(k_1, ..., k_n) = \sum_{j=1}^{n} \frac{1}{\sqrt{n}} \delta_{kk_j} \Psi_{n-1}(k_1, ..., k_{j-1}, k_{j+1}, ..., k_n),$$

$$(\hat{a}_k^- \Psi)_{n-1}(k_1, ..., k_{n-1}) = \sqrt{n}\Psi_n(k, k_1, ..., k_{n-1}).$$

The operators $\hat{V}_i$ act in the space of functions $\psi(V_0, ..., V_{l-1})$ from $L^2(\mathbf{R}^l)$ as operators of multiplication by V_i, while $\hat{P}_i = -i\frac{\partial}{\partial V_i}$.

Choose, as usual, the space $\mathcal{F} \otimes L^2(\mathbf{R}^l)$ as a state space of the composed system; the operators $\hat{P}_i$, $\hat{V}_i$ and $\hat{a}_k^{\pm}$ are extended as $\hat{P}_i \otimes 1$, $\hat{V}_i \otimes 1$ and $1 \otimes \hat{a}_k^{\pm}$ correspondingly.

Consider eq.(10) for this model. Instead of the operator $e^{iH_0 t}(\hat{H} - \hat{H}_0)e^{-iH_0 t}$ entering to the right-hand side of eq.(10), where $\hat{H}_0 = \hat{H}_Q + \hat{H}_q$, consider the operator of the form

$$e^{i\hat{H}_q t}(\hat{H} - \hat{H}_q)e^{-i\hat{H}_q t} = \hat{H}_Q + \hat{H}_{ct}^{\Lambda} + g\sum_{k=1}^{\infty} \mu_k^{\Lambda} \hat{Q} \frac{\hat{a}_k^+ e^{i\Omega_k t} + \hat{a}_k^- e^{-i\Omega_k t}}{\sqrt{2\Omega_k}}. \tag{17}$$

Denote by $U_{t_1 t_2}^{\Lambda}$ the evolution operator for the Hamiltonian (17) transforming the initial condition at for the equation

$$i\frac{d}{dt}\Phi_{\Lambda}^t = e^{i\hat{H}_q t}(\hat{H} - \hat{H}_q)e^{-i\hat{H}_q t}\Phi_{\Lambda}^t$$

at $t = t_1$ to the solution of this equation at $T = t_2$, $\Phi_{\Lambda}^{t_2} = U_{t_1 t_2}^{\Lambda} \Phi_{\Lambda}^{t_1}$. The evolution operator $\tilde{U}_{t_1 t_2}^{\Lambda}$ for eq.(10) is related with $U_{t_1 t_2}^{\Lambda}$ as

$$\tilde{U}_{t_1 t_2}^{\Lambda} = e^{iH_Q t_2} U_{t_1 t_2}^{\Lambda} e^{-iH_Q t_1}.$$

Let $g = g(t) = g_t$ be a function vanishing at $|t| > T$. Then the Bogoliubov S-matrix coincides with the operator $\tilde{U}_{-TT}^{\Lambda}$ and can be expressed then via the evolution operator U_{-TT}^{Λ} for the Hamiltonian (17). We will show that this operator has a strong limit as $\Lambda \to \infty$. This will imply that the Bogoliubov S-matrix and the renormalized evolution operator for the $g = const$-case are well-defined as $\Lambda = \infty$.

2.2.2 Renormalization of divergences in classical equations

It is well-known [8, 14] that quantum theories with quadratic Hamiltonians are specified by their classical analogs. Consider the divergences in the classical version of the model (17). Equations of motion for the quantum Heisenberg operators coincide with the classical equations and have the form:

$$i\dot{a}_k^- = g\frac{\mu_k}{\sqrt{2\Omega_k}}Qe^{i\Omega_k t}, \qquad -i\dot{a}_k^+ = g\frac{\mu_k}{\sqrt{2\Omega_k}}Qe^{-i\Omega_k t}; \tag{18}$$

$$\begin{array}{c} \dot{V}_j = V_{j+1}, j = \overline{0, l-1}; \dot{V}_{l-1} = (-1)^{l-1}\frac{P_{l-1}}{z_l}; \\ -\dot{P}_j = (-1)^j(z_j + \delta z_j)V_j + P_{j-1}, j = \overline{1, l-1}; \\ -\dot{P}_0 = (z_0 + \delta z_0)Q + g\sum_{k=1}^{\infty}\mu_k\frac{a_k^+ e^{i\Omega_k t} + a_k^- e^{-i\Omega_k t}}{\sqrt{2\Omega_k}}. \end{array} \tag{19}$$

Eqs.(19) can be presented as

$$\sum_{s=0}^{l}(z_s Q^{(s)})^{(s)} + \sum_{s=0}^{l-1}(\delta z_s Q^{(s)})^{(s)} + g\sum_{k=1}^{\infty}\mu_k\frac{a_k^+ e^{i\Omega_k t} + a_k^- e^{-i\Omega_k t}}{\sqrt{2\Omega_k}} = 0, \tag{20}$$

while integration of eqs.(18) gives us the following relations:

$$a_k^{\pm}(t) = a_k^{\pm}(-\infty) \pm i\frac{\mu_k}{\sqrt{2\Omega_k}}\int_{-\infty}^{t} d\tau g_\tau Q_\tau e^{\mp i\Omega_k \tau}. \tag{21}$$

After substitution of formula (21) and integration by parts $2l$ times eq. (20) is transformed to the following form:

$$\begin{array}{c} \sum_{s=0}^{l}(z_s Q_t^{(s)})^{(s)} + \sum_{k=1}^{\infty} g_t\mu_k\frac{a_k^+(-\infty)e^{i\Omega_k t} + a_k^-(-\infty)e^{-i\Omega_k t}}{\sqrt{2\Omega_k}} \\ +(-1)^{l+1}g_t\sum_{k=1}^{\infty}\frac{\mu_k^2}{\Omega_k}\int_{-\infty}^{t}d\tau(g_\tau Q_\tau)^{(2l)}\frac{sin(\Omega_k(t-\tau))}{\Omega_k^{2l}} = -(\hat{B}_1 + \hat{B}_2)Q(t) \end{array} \tag{22}$$

where

$$\hat{B}_1 = \sum_{s=0}^{l-1}\frac{\mu_k^2}{\Omega_k^{2s+2}}(-1)^{s+1}g_t\frac{d^{2s}}{dt^{2s}}g_t; \qquad \hat{B}_2 = \sum_{s=0}^{l-1}\frac{d^s}{dt^s}\delta z_s\frac{d^s}{dt^s}.$$

Check that under some choice of the counterterms $\hat{B}_2 + \hat{B}_1 = 0$. Note that the operator $\hat{B}_1$ is Hermitian and polynomial in d/dt. The degree of the polynomial is $2l-2$, so that $\hat{B}_1 = \sum_{j=0}^{2l-2} b_j(t)\frac{d^j}{dt^j}$. Choose

$$\hat{B}_2^{(1)} = \frac{d^{l-1}}{dt^{l-1}}b_{2l-2}(t)\frac{d^{l-1}}{dt^{l-1}} = b_{2l-2}\frac{d^{2l-2}}{dt^{2l-2}} + (l-1)\dot{b}_{2l-2}\frac{d^{2l-3}}{dt^{2l-3}} + ...$$

One has

$$\hat{B}_1^+ = b_{2l-2}\frac{d^{2l-2}}{dt^{2l-2}} + ((2l-2)\dot{b}_{2l-2} - b_{2l-3})\frac{d^{2l-3}}{dt^{2l-3}} + ...$$

and $b_{2l-3} = (l-1)\dot{b}_{2l-2}$, so that the operator $\hat{B}_1 - \hat{B}_2^{(1)}$ contains the derivative d/dt in degrees no higher than $2l-4$. Applying to the operator $\hat{B}_1 - \hat{B}_2^{(1)}$ an analogous procedure $l-2$ times, one constructs such operator $\hat{B}_2 = \hat{B}_2^{(1)} + ... + \hat{B}_2^{(l-2)}$, that $\hat{B}_2 = -\hat{B}_1$.

Eq.(22) does not contain singularities if

$$\sum_k \frac{\mu_k^2}{\Omega_k^{2l+1}} < \infty. \tag{23}$$

Existence, uniqueness and smoothness of the solution of the Cauchy problem for this equation are corollaries of the general theory of integral equations (see, for example, [36]). Thus, the classical theory does not contain singularities as $\Lambda \to \infty$, provided that the considered counterterms are added and condition (23) is satisfied.

2.2.3 Regularity of the Bogoliubov S-matrix

Let us check that under condition (23) the evolution operator U^{Λ}_{-TT} for the time interval $(-T,T)$ is regular as $\Lambda\to\infty$ in the theory with Hamiltonian (17). For simplicity of the notations, the index Λ will be omitted. Represent the operators $\hat{Q}=V_0$, $\hat{V}_s$, $\hat{P}_s$ via creation and annihilation operators $\hat{B}^{\pm}_k=\frac{\hat{V}_k\mp i\hat{P}_k}{\sqrt{2}}$; denote $\hat{B}^{\pm}_{l+k}=\hat{a}^{\pm}_k$. The Hamiltonian (17) is quadratic in creation-annihilation operators:

$$H=\sum_{ij=1}^{\infty}\left[\frac{1}{2}\hat{B}^+_iR_{ij}\hat{B}^+_j+\hat{B}^+_iT_{ij}\hat{B}^-_j+\frac{1}{2}\hat{B}^-_iR^*_{ij}\hat{B}^-_j\right]+\varepsilon_0.$$

Consider the linear canonical Bogoliubov transformation, transforming the initial condition $B^{\pm}_k(-T)$ for the set of equations (18)-(19) to the solution of this set $B^{\pm}_k(T)$ at $t=T$. The Wick symbol of the evolution operator $U^{\Lambda}_{-TT}=: U_{\Lambda}(B^+,B^-):$ for this theory is presented as [8]:

$$\begin{gathered}U(B^*,B)=\frac{\exp(i\int_{-T}^{T}d\tau[\frac{1}{2}\sum_{i=1}^{\infty}T_{ii}-\varepsilon_0])}{\sqrt{detG_{\Lambda}}}\\ \times\exp\sum_{ij=1}^{\infty}\left[-\tfrac{1}{2}B_i(G_{\Lambda}^{-1}F^*_{\Lambda})_{ij}B_j+B_i(G_{\Lambda}^{-1}-1)_{ij}B^*_j+\tfrac{1}{2}B^*_i(F_{\Lambda}G_{\Lambda}^{-1})_{ij}B^*_j\right]\end{gathered}\qquad(24)$$

where

$$(F_{\Lambda})_{ij}=\frac{\partial B^-_i(T)}{\partial B^+_j(-T)},\ (G_{\Lambda})_{ij}=\frac{\partial B^+_i(T)}{\partial B^+_j(-T)}.$$

According to the appendix, the conditions

$$\begin{gathered}\sum_{ij}|(G_{\Lambda})_{ij}-G_{ij}|^2\to_{\Lambda\to\infty}0,\sum_{ij}|(F_{\Lambda})_{ij}-F_{ij}|^2\to_{\Lambda\to\infty}0,\\ detG_{\Lambda}\to_{\Lambda\to\infty}detG\end{gathered}\qquad(25)$$

imply that the operator U^{Λ}_{-TT} has a strong limit as $\Lambda\to\infty$. To check condition (25), it is necessary to show that

(a) the l^2-vectors of the form $\frac{\partial Q^{(s)}(T)}{\partial a^+_k(-T)}$ and $\frac{\partial a^+_k(T)}{\partial Q^{(s)}(-T)}$ have strong limits as $s=0,...,2l-1$.

(b) the operators with matrices $\frac{\partial a^+_k(T)}{\partial a^{\pm}_m(-T)}$ are presented as

$$\frac{\partial a^+_k(T)}{\partial a^-_m(-T)}=(A^{(1)}_{\Lambda}A^{(2)}_{\Lambda}A^{(3)}_{\Lambda})_{km},\ \frac{\partial a^+_k(T)}{\partial a^+_m(-T)}=(1+A^{(4)}_{\Lambda}A^{(5)}_{\Lambda}A^{(6)}_{\Lambda})_{km},$$

for some operators $A^{(i)}$ that converge as $\Lambda\to\infty$ in the norm

$$||A||_2=\sqrt{TrA^+A}.\qquad(26)$$

Namely, under conditions (a) and (b) the operators F_{Λ} and G_{Λ} are evidently converge as $\Lambda\to\infty$, while the determinant $detG_{\Lambda}$ converges because of lemma 2 of the appendix.

The matrices are expressed via the fundamental solution of the equation

$$\sum_{s=0}^{2l}(z_sQ_t^{(s)})^{(s)}+(-1)^{l+1}g_t\int_{-\infty}^{t}d\tau(g_{\tau}Q_{\tau})^{(2l)}\sum_{k=1}^{\infty}\frac{\mu_k^2}{\Omega_k^{2l+1}}\sin[\Omega_k(t-\tau)]=j_t.\qquad(27)$$

The solution of eq.(27) is expressed via the linear combination of the initial conditions and the right-hand side:

$$Q_t=\sum_{s=0}^{2l-1}c_s(t)Q^{(s)}(-T)+\int_{-\infty}^{t}d\tau G_{t\tau}j_{\tau}.$$

Let the function $G_{t\tau}$ be equal to zero at $t<\tau$. Then the integration can be supposed to be taken from $-\infty$ to $+\infty$.

It follows from eqs.(21) and (22) that

$$\begin{aligned} \frac{\partial Q^{(s)}(T)}{\partial a_k^+(-T)} &= -\int_{-\infty}^{\infty} d\tau \frac{\partial^s}{\partial T^s}\frac{\partial^l}{\partial \tau^l}(G_{T\tau}g_\tau)\frac{i^l\mu_k e^{i\Omega_k\tau}}{\sqrt{2}\Omega_k^{l+1/2}}, \\ \frac{\partial a_k^+(T)}{\partial Q^{(s)}(-T)} &= i^{l+1}\int_{-\infty}^{\infty} d\tau \frac{\partial^l}{\partial\tau^l}(g_\tau c_s(\tau))\frac{\mu_k e^{i\Omega_k\tau}}{\sqrt{2}\Omega_k^{l+1/2}}, \end{aligned} \tag{28}$$

$$\begin{aligned} \frac{\partial a_k^+(T)}{\partial a_m^+(-T)} &= \delta_{km} - i(-1)^l\int d\tau_1 d\tau_2 \frac{\mu_k e^{i\Omega_k\tau_1}}{\sqrt{2}\Omega_k^{l+1/2}}\frac{\mu_m e^{i\Omega_k\tau_2}}{\sqrt{2}\Omega_m^{l+1/2}}\frac{\partial^l}{\partial\tau_1^l}\frac{\partial^l}{\partial\tau_2^l}(g_{\tau_1}G_{\tau_1\tau_2}g_{\tau_2}) \\ \frac{\partial a_k^+(T)}{\partial a_m^-(-T)} &= -i\int d\tau_1 d\tau_2 \frac{\mu_k e^{i\Omega_k\tau_1}}{\sqrt{2}\Omega_k^{l+1/2}}\frac{\mu_m e^{-i\Omega_k\tau_2}}{\sqrt{2}\Omega_m^{l+1/2}}\frac{\partial^l}{\partial\tau_1^l}\frac{\partial^l}{\partial\tau_2^l}(g_{\tau_1}G_{\tau_1\tau_2}g_{\tau_2}) \end{aligned} \tag{29}$$

To justify the properties (a) and (b), it is sufficient to prove the uniform convergence of the functions

$$\frac{\partial^s}{\partial T^s}\frac{\partial^l}{\partial\tau^l}(G_{T\tau}g_\tau), \quad \frac{\partial^l}{\partial\tau^l}(g_\tau c_s(\tau)), \quad \frac{\partial^l}{\partial\tau_1^l}\frac{\partial^l}{\partial\tau_2^l}(g_{\tau_1}G_{\tau_1\tau_2}g_{\tau_2}) \tag{30}$$

at $[-T,T]$. Namely, the property (23) implies the convergence of vectors (28). Construct operators $A^{(i)}$. The operator $A^{(3)}$ transforms the sequence f_m from l^2 to the function $(A^{(3)}f)(\tau) = \sum_m \frac{\mu_m e^{-i\Omega_m\tau}}{\sqrt{2}\Omega_m^{l+1/2}}f_m$, let $A^{(2)}$ be an integral operator with the kernel $\frac{\partial^l}{\partial\tau_1^l}\frac{\partial^l}{\partial\tau_2^l}(g_{\tau_1}G_{\tau_1\tau_2}g_{\tau_2})$, while the operator $A^{(1)}$ transforms the function φ from $L^2[-T,T]$ to the sequence $(A^{(1)}\varphi)_m = -i\int d\tau \frac{\mu_m e^{i\Omega_m\tau}}{\sqrt{2}\Omega_m^{l+1/2}}f_m$. Choose $A^{(5)} = A^{(2)}$, $A^{(4)} = A^{(1)}$, $(A^{(6)}f)(\tau) = \sum_m \frac{(-1)^l\mu_m e^{i\Omega_m\tau}}{\sqrt{2}\Omega_m^{l+1/2}}f_m$. Convergence of these operator in the (26)-norm is a corollary of the property (23) and convergence of the functions (30).

Convergence of the function $\frac{\partial^l}{\partial\tau^l}(g_\tau c_s(\tau))$ is a corollary of the lemma 3 of the appendix. To investigate the property of convergence of the function $G^{(l)(l)} \equiv \frac{\partial^l}{\partial\tau_1^l}\frac{\partial^l}{\partial\tau_2^l}G_{\tau_1\tau_2}$, represent eq.(27) in the form

$$\begin{aligned} \sum_{s=0}^l z_s\left(\frac{d}{dt}\right)^{2s-2l}G^{(l)(l)}_{tt_0} + (-1)^{l+1}\left(\frac{d}{dt}\right)^{-l}g\left(\frac{d}{dt}\right)^l\hat{K}\left(\frac{d}{dt}\right)^l g\left(\frac{d}{dt}\right)^{-l}G^{(l)(l)}_{tt_0} \\ = (-1)^l\delta(t-t_0), \end{aligned}$$

where $\left(\frac{d}{dt}\right)^{-1}$ is an integral operator
$(\left(\frac{d}{dt}\right)^{-1}f)(t) = \int_{-\infty}^t f(\tau)d\tau$, while $\hat{K}$ is the operator with the kernel

$$K(t,\tau) = \sum_{k=1}^{\infty}\frac{\mu_k^2}{\Omega_k^{2l+1}}\sin[\Omega_k(t-\tau)].$$

Convergence of functions $G^{(l)(l)}$ and $\frac{\partial^m}{\partial t^m}\frac{\partial^n}{\partial\tau^n}G_{t\tau}$, $m,n\le l$, is a corollary of lemma 3.

As $t>T>\tau$, the functions $G_{t\tau}$ obey the equation $\sum_{s=0}^{2l} z_s\frac{\partial^s}{\partial t^{2s}}G_{t\tau} = 0$. Therefore

$$\begin{aligned} G_{tt_0} = \sum_{s=0}^{2l-1}\frac{\partial^s}{\partial T^s}G_{Tt_0}\sum_{m=[\frac{s}{2}+1]}^{l}\frac{(t-T)^{2l-2m+s}}{(2l-2m+s)!}\frac{z_m}{z_l} \\ -\sum_{m=0}^{l-1}\int_T^t d\tau G_{\tau t_0}\frac{z_m}{z_l}\frac{(t-\tau)^{2l-1-2m}}{(2l-1-2m)!} \end{aligned} \tag{31}$$

The quantity $\frac{\partial^s}{\partial T^s}G_{Tt_0}$ is expressed via linear combinations of the values $G_{\tau t_0}$ at $\tau\in(T,T+\delta)$ from eq.(31) for $t=t_1,...,t_{2s}\in(T,T+\delta)$. Therefore, the convergence of functions $\frac{\partial^s}{\partial T^s}\frac{\partial^m}{\partial\tau^m}G_{T\tau}$ is a corollary of convergence of the quantity $\frac{\partial^m}{\partial\tau^m}G_{T\tau}$.

Thus, the strong convergence of the Bogoliubov S-matrix as $\Lambda\to\infty$ is checked for smooth functions g which vanish for $t\notin(-T,T)$. Convergence of the operator S^+ is checked analogously.

2.3 Representations of different operators

In this subsection we investigate what operators O_Λ transform the vectors of the type $T_\Lambda\Psi_1$ to the vectors of the same type $T_\Lambda\Psi_2$. This means that the condition (6) is invariant under transformation O_Λ. In this case the operator (8) is regular as $\Lambda\to\infty$.

Consider the Heisenberg operators $O_\Lambda = \hat{a}_k^+(t) = e^{iHt}\hat{a}_k^+e^{-iHt}$. Since the operators e^{-iHt} and T_Λ can be expressed via the evolution operator $V_{t_1t_2}$ for the theory with the Hamiltonian (17),

$$e^{-iHt} = e^{-iH_qt}U_{0t}, \qquad T_\Lambda = U_{-T0}e^{iH_QT}$$

for Heisenberg operators $\hat{a}_k^+(t)$ in the representation (8) one has

$$\begin{aligned} T_\Lambda^+\hat{a}_k^+(t)T_\Lambda &= e^{-iH_QT}U^+_{-Tt}e^{iH_qt}\hat{a}_k^+e^{-iH_qt}U_{-Tt}e^{iH_QT} \\ &= e^{-iH_QT}U^+_{-Tt}\hat{a}_k^+U_{-Tt}e^{iH_QT}e^{i\Omega_kt}. \end{aligned}$$

Analogously, one has

$$T_\Lambda^+\hat{a}_k^-(t)T_\Lambda = e^{-iH_QT}U^+_{-Tt}\hat{a}_k^-U_{-Tt}e^{iH_QT}e^{-i\Omega_kt}.$$

$$T_\Lambda^+\hat{Q}(t)T_\Lambda = e^{-iH_QT}U^+_{-Tt}\hat{Q}U_{-Tt}e^{iH_QT}e^{-i\Omega_kt}.$$

To investigate the regularity of the operators (8), it is sufficient to investigate the regularity as $\Lambda\to\infty$ of the operators

$$V^+_{-Tt}\hat{a}_k^\pm V_{-Tt}, \qquad V^+_{-Tt}\hat{Q}V_{-Tt}. \tag{32}$$

The Heisenberg equations of motion for the operators (32) coincide with the classical equations in the model (17). Therefore, for operators (32) one has

$$\begin{aligned} U^+_{-Tt}\hat{a}_k^\pm U_{-Tt} &= \sum_{m=1}^\infty \left(\frac{\partial a_k^\pm(t)}{\partial a_m^-(-T)}\hat{a}_m^- + \frac{\partial a_k^\pm(t)}{\partial a_m^+(-T)}\hat{a}_m^+ \right) + \sum_{s=0}^{2l-1}\frac{\partial a_k^\pm(t)}{\partial Q^{(s)}(-T)}\hat{Q}^{(s)} \\ U^+_{-Tt}\hat{Q}U_{-Tt} &= \sum_{m=1}^\infty \left(\frac{\partial Q(t)}{\partial a_m^-(-T)}\hat{a}_m^- + \frac{\partial Q(t)}{\partial a_m^+(-T)}\hat{a}_m^+ \right) + \sum_{s=0}^{2l-1}\frac{\partial Q(t)}{\partial Q^{(s)}(-T)}\hat{Q}^{(s)}. \end{aligned} \tag{33}$$

It follows from the previous subsection that the operators $T_\Lambda^+a_k^\pm T_\Lambda$ and $T_\Lambda^+Q^{(s)}T_\Lambda$, $s=\overline{0,l}$ are regular as $\Lambda\to\infty$. Equations of motion imply that the operators $T_\Lambda^+P_sT_\Lambda$, $s=\overline{0,l-1}$, are singular as $\Lambda\to\infty$.

3 Non-unitarity of evolution in the external field

It will be shown in this section that the evolution operator corresponding to the model (24) in the external field in the leading order of perturbation theory may be nonunitary, since it is necessary to consider different representations of the canonical commutation relations at different moments of time.

The quantum theory in the external field is usually constructed as follows [6]. The field $Q(t)$ is decomposed into two parts. The "classical" part $\frac{1}{g}Q_c(t)$ is of order $O(1/g)$. The remaining part $\hat{X}(t)$ is "quantum",

$$Q(t) = \frac{1}{g}Q_c(t) + \hat{X}(t). \tag{34}$$

The classical part of the field q_k will be set to zero. Action (24) takes the following form:

$$S = const - \frac{1}{g}\int dt\hat{X}\sum_{s=0}^{l} z_s Q_c^{(2s)} + \int dt[L_X + \sum_{k=1}^{\infty}\left(\frac{\dot{q}_k^2}{2} - \frac{\Omega_k^2 q_k^2}{2}\right) - \sum_{k=1}^{\infty}\mu_k q_k Q_c] + O(g).$$

The term of order $1/g$ vanishes if the "external field" $Q_c(t)$ obeys the classical equation of motion

$$\sum_{s=0}^{l} z_s Q_c^{(2s)} = 0,$$

which can be obtained from eq.(24) by the variation procedure as $g \to 0$. Neglect the terms of order $O(g)$. We obtain that the degrees of freedom corresponding to fields $\hat{X}$ and q_k are independent. Thus, one can consider the problem of quantization of the fields q_k in the external nonstationary classical field $Q_c(t)$. The Hamiltonian of this model has the form:

$$H = \sum_{k=1}^{\infty}\left(\frac{p_k^2}{2} + \frac{\Omega_k^2 q_k^2}{2}\right) + \sum_{k=1}^{\infty}\mu_k q_k Q_c(t). \tag{35}$$

3.1 Fock representation: range of validity

One can try to use the Fock representation of the canonical commutation relations (15). Let us investigate in what case it is possible.

Under this choice of the representation, the Hamiltonian (35) takes the form:

$$H = \sum_k \Omega_k a_k^+ a_k^- + \sum_{k=1}^{\infty}\mu_k \frac{a_k^+ + a_k^-}{\sqrt{2\Omega_k}} Q_c + E_0 \tag{36}$$

If we choose the Wick ordering of creation and annihilation operators, the constant E_0 vanishes.

Consider the solution to the Schrodinger equation

$$i\frac{d\Psi}{dt} = H\Psi, \tag{37}$$

which has the form of the coherent state

$$\Psi(t) = c(t)\exp[\sum_{k=1}^{\infty}\alpha_k(t)\hat{a}_k^+]|0>, \tag{38}$$

being expressible via the vacuum vector $|0>$ of the form $(1,0,0,...)$ and complex functions $c(t)$ and $\alpha_k(t)$. Substitution of the vector (38) to eq.(37) leads us to the relations

$$i\dot{c} = \left[\sum_{k=1}^{\infty}\frac{\mu_k\alpha_k}{\sqrt{2\Omega_k}}\mathcal{Q} + E_0\right]c; \qquad i\dot{\alpha}_k = \Omega_k\alpha_k + \frac{\mu_k}{\sqrt{2\Omega_k}}\mathcal{Q}.$$

The divergences appearing in the multiplier c can be eliminated by the proper choice of the "counterterm" E_0. Investigate now the functions $\alpha_k(t)$:

$$\alpha_k(t) = \alpha_k(0)e^{-i\Omega_k t} + \rho_k(t), \tag{39}$$

where

$$\rho_k(t) = -i\frac{\mu_k}{\sqrt{2\Omega_k}}\int_0^t d\tau\mathcal{Q}(\tau)e^{-i\Omega_k(t-\tau)}. \tag{40}$$

It follows from [8] that expression (38) defines the Fock vector if

$$\sum_k |\alpha_k(t)|^2 < \infty. \tag{41}$$

Integrating eq.(40) by parts, we obtain

$$\rho_k(t) = \beta_k^l(t) - \beta_k^l(0)e^{-i\Omega_k t} + \gamma_k^l(t), \tag{42}$$

where

$$\beta_k^l(t) = -\frac{\mu_k}{\sqrt{2\Omega_k}} \sum_{s=0}^{l-2} i^s \mathcal{Q}^{(s)}(t)\frac{1}{\Omega_k^{s+1}};$$
$$\gamma_k^l(t) = -\frac{\mu_k}{\sqrt{2\Omega_k}} \int_0^t i^l d\tau \mathcal{Q}^{(l-1)}(\tau)\frac{e^{i\Omega_k(\tau-t)}}{\Omega_k^{l-1}}.$$

The leading in $1/\Omega_k$ order is $\rho_k(t) \simeq -\frac{\mu_k}{\sqrt{2\Omega_k^3}}(\mathcal{Q}(t) - \mathcal{Q}(0)e^{-i\Omega_k t})$. Condition (41) is satisfied if

$$\sum_{k=1}^{\infty} \frac{\mu_k^2}{\Omega_k^3} < \infty. \tag{43}$$

Thus, the evolution transformation in the external field can be viewed as an unitary operator if the condition (43) is satisfied.

Condition (43) can be also obtained as follows. Heisenberg equations of motion for the operators $\pi(a_k^{\pm}(t)) = e^{i\hat{H}t}\pi(a_k^{\pm})e^{-i\hat{H}t}$ are written as

$$\mp i\frac{d}{dt}\pi(a_k^{\pm}(t)) = \Omega_k \pi(a_k^{\pm}(t)) + \frac{\mu_k}{\sqrt{2\Omega_k}}\mathcal{Q}(t). \tag{44}$$

Heisenberg creation and annihilation operators at different time moments are related as

$$\pi(a_k^{\pm}(t)) = e^{-i\Omega_k t}\pi(a_k^{\pm}(0)) + \rho_k^{\pm}(t), \tag{45}$$

where $\rho_k^- = \rho_k$, $\rho_k^+ = \rho_k^*$, $\pi(a_k^{\pm}(0)) = \hat{a}_k^{\pm}$. According to [8], the canonical transformation (45) is unitary if and only if $\sum_k |\rho_k(t)|^2 < \infty$. This condition is equivalent to (43).

3.2 Different Hilbert spaces at different time moments

If the condition (43) is not satisfied, one should consider non-Fock representations of CCR in order to construct the quantum theory. Since the choice of the non-Fock representation is specified by the interaction (see, for example, [27]), which depends on time in our case, it is necessary to consider different representations of CCR at different time moments.

Consider this hypothesis for the model (36).

One can consider the "large" linear state space $\mathcal{L}$ and specify the subspaces $\mathcal{H}_\alpha \subset \mathcal{L}$. The inner product is introduced on each subspace $\mathcal{H}_\alpha$. The parameter α belongs to some set A. The operators $e^{i\sum_k \hat{a}_k^+ z_k}$ defined on $\mathcal{L}$ transform elements of $\mathcal{H}_\alpha$ to elements of $\mathcal{H}_\alpha$. The restrictions $\pi_\alpha(a_k^{\pm}) = \hat{a}_k^{\pm}|_{\mathcal{H}_\alpha}$ of creation and annihilation operators on the subspace $\mathcal{H}_\alpha$ specifies the α-representation of CCR.

The evolution operator U_t is defined as a set of mappings $u_t : A \to A$ and $V_t^\alpha : \mathcal{H}_\alpha \to \mathcal{H}_{u_t\alpha}$. If the initial condition Ψ_0 belongs to $\mathcal{H}_\alpha$, the state at time moment t is defined as $V_t^\alpha \Psi_0$ and belongs to $\mathcal{H}_{u_t\alpha}$.

Choose as a space $\mathcal{L}$ the space of analytic functionals $\Psi(z) = \Psi(z_1, z_2, ...)$. The creation and annihilation operators have the form:

$$\hat{a}_k^+ = z_k, \qquad \hat{a}_k^- = \frac{\partial}{\partial z_k}. \tag{46}$$

Define the subset $\mathcal{H}_0 \in \mathcal{L}$ as follows. Consider the expansion of the functional Ψ into a series:

$$\Psi = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \sum_{i_1...i_n} \Psi^{(n)}_{i_1...i_n} z_{i_1}...z_{i_n}$$

with symmetric in $i_1...i_n$ coefficient functions $\Psi^{(n)}_{i_1...i_n}$. We say that $\Psi \in \mathcal{H}_0$ if

$$||\Psi||^2 = \sum_{n=0}^{\infty} \sum_{i_1...i_n} |\Psi^{(n)}_{i_1...i_n}|^2 < \infty. \quad (47)$$

Introduce the inner product on $\mathcal{H}_0$:

$$(\tilde{\Psi}, \Psi)_0 = \sum_{n=0}^{\infty} \tilde{\Psi}^{(n)*}_{i_1...i_n} \Psi^{(n)}_{i_1...i_n}.$$

Note that the space $\mathcal{H}_0$ is isomorphic to the Fock space [8].

Let $\alpha = (\alpha_1, \alpha_2, ...)$ is a set of complex umbers. Say that $\Psi \in \mathcal{H}_\alpha$ if the functional

$$w_\alpha \Psi(z) = e^{-\sum_{k=1}^{\infty} \alpha_k (z_k + \alpha_k^*)} \Psi(z + \alpha^*)$$

belongs to $\mathcal{H}_0$. Introduce on $\mathcal{H}_\alpha$ the inner product:

$$(\tilde{\Psi}, \Psi)_\alpha = (w_\alpha \tilde{\Psi}, w_\alpha \Psi)_0$$

In particular, the functional $\Psi(z) = e^{\sum_{k=1}^{\infty} z_k \alpha_k}$ belongs to $\mathcal{H}_\alpha$ in any case. This functional belongs to $\mathcal{H}_0$ if and only if $\alpha \in l^2$.

Let $\Psi_0 \in \mathcal{H}_{\alpha(0)}$. Define the mapping $u_t : \alpha(0) \mapsto \alpha(t)$ according to (39). Since the evolution equation in the representation (46) has the form

$$i\dot{\Psi}^t(z) = \left(\sum_{k=1}^{\infty} \Omega_k z_k \frac{\partial}{\partial z_k} + \sum_{k=1}^{\infty} \mu_k \frac{z_k + \frac{\partial}{\partial z_k}}{\sqrt{2\Omega_k}} Q_c + E_0 \right) \Psi^t(z).$$

The functional $\Phi^t = w_{\alpha(t)} \Psi^t$ obeys the equation:

$$i\dot{\Phi}^t(z) = \left(\sum_{k=1}^{\infty} \Omega_k z_k \frac{\partial}{\partial z_k} + \sum_{k=1}^{\infty} \mu_k \frac{\alpha_k}{\sqrt{2\Omega_k}} Q_c + E_0 \right) \Phi^t(z).$$

If $\Phi^0 \in \mathcal{H}_0$, one has $\Phi^t(z) = b^t \Phi^0(z e^{-i\Omega t}))$ for some multiplier c^t. Under the appropriate choice of the counterterm E_0, $\Phi^t \in \mathcal{H}_0$. Thus, $\Psi^t \in \mathcal{H}_{\alpha(t)}$.

3.3 An algebraic approach

The lack of the approach of the previous subsection is that it is necessary to eliminate the divergences from the multiplier b^t by renormalization of E_0. If one considered the density matrix instead of the wave function, this difficulty does not arise. The generalization of the density-matrix approach to systems of infinite number of degrees of freedom is the algebraic approach [27] which is suitable for the case when different representations of CCR arise at different time moments.

3.3.1 Density matrix for systems of infinite number of degrees of freedom

In the d-dimensional quantum mechanics, one can use not only the wave-function language but also the density-matrix conception. If the system is in a pure state with the wave function Ψ, the Blokhintsev-Wigner density matrix is determined as:

$$\rho(p, q) = \frac{1}{(2\pi)^d} \int d\xi e^{ip\xi} \Psi(q - \frac{\xi}{2}) \Psi^*(q + \frac{\xi}{2}). \quad (48)$$

A remarkable property of the density (48) is that the average values of observables $\hat{A} = A(\hat{p}, \hat{q})$ can be presented in a form analogous to the classical statistical mechanics:

$$< \hat{A} >= (\Psi, \hat{A}\Psi) = \int dpdq A(p,q)\rho(p,q),$$

provided that the Weyl ordering of the coordinate and momenta operators are chosen. For pure states, the density matrix specifies the wave function Ψ up to a multiplier.

For the case of infinite number of degrees of freedom, the numerous difficulties with the divergences arise. Nevertheless, one can consider the Fourier transformation of ρ:

$$\tilde{\rho}(\alpha, \beta) = \int dpdq \rho(p,q) e^{-i\alpha p - i\beta q}.$$

For pure states (48), it can be presented as

$$\hat{\rho}(\alpha, \beta) = (\Psi, e^{-i\alpha\hat{p} - i\beta\hat{q}}\Psi). \tag{49}$$

The function (49) can be used instead of the density matrix in calculations of the average values of observables.

The advantage of using the function (49) is the possibility of generalization to the infinite-dimensional case. One can specify the state of the system by the average values

$$\tilde{\rho}(z, z^*) =< e^{\sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)} > . \tag{50}$$

Consider some examples of "densities" (50).

1. For the vacuum state

$$\tilde{\rho}(z, z^*) = (\Phi^{(0)} e^{\sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)} \Phi^{(0)}) = e^{-\frac{1}{2}\sum_k z_k^* z_k}.$$

2. For the coherent state

$$\Phi = e^{\sum_k (\alpha_k \hat{a}_k^+ - \alpha_k^* \hat{a}_k^-)} \Phi^{(0)},$$

where $\alpha \in l^2$, one has

$$\tilde{\rho}(z, z^*) = e^{\sum_k (\alpha_k^* z_k - \alpha_k z_k^* - \frac{1}{2} z_k^* z_k)}. \tag{51}$$

3. Suppose that the non-Fock representation $\pi(a_l^{\pm})$ of CCR in the space $\mathcal{H}$ is chosen. For this case, one can also use the "density" (50):

$$\tilde{\rho}(z, z^*) = (\Phi e^{\sum_k (z_k \pi(a_k^+) - z_k^* \pi(a_k^-))} \Phi), \qquad \Phi \in \mathcal{H}. \tag{52}$$

4. As an example, consider the following "α- representation" of CCR:

$$\mathcal{H} = \mathcal{F}, \qquad \pi_\alpha(a_k^+) = \hat{a}_k^+ + \alpha_k^*, \qquad \pi_\alpha(a_k^-) = \hat{a}_k^- + \alpha_k. \tag{53}$$

For the vacuum state $\Phi = \Phi^{(0)}$, the "density" $\tilde{\rho}$ has the form (51), but the case $\alpha \in l^2$ can be involved.

Definition. We say that the function $\rho(z, z^*)$ is an α-*density* if it is written in the form (52) for the representation (53). For $\alpha = 0$, α-densities will be called as Fock densities.

Statement 1. $\tilde{\rho}(z, z^*)$ is an α-density if and only if $\tilde{\rho}(z, z^*) e^{\sum_{k=1}^\infty (-\alpha_k^* z_k + \alpha_k z_k^*)}$ is a Fock density.

Proof. The fact that $\tilde{\rho}(z, z^*)$ is an α-density means that

$$\tilde{\rho}(z, z^*) = (\Phi e^{\sum_k (z_k(\hat{a}_k^+ + \alpha_k^*) - z_k^*(\hat{a}_k^- + \alpha_k))} \Phi),$$

for some vector $\Phi \in \mathcal{F}$. This is equivalent to the statement that $\tilde{\rho}(z,z^*)e^{\sum_{k=1}^\infty(-\alpha_k^* z_k + \alpha_k z_k^*)}$ is a Fock density.

Statement 2. Let $\tilde{\rho}$ be an $\alpha^{(1)}$-density. Then $\tilde{\rho}$ is an $\alpha^{(2)}$-density if and only if $\alpha^{(2)} - \alpha^{(1)} \in l^2$.

Proof. According to statement 1, the function

$$\tilde{\rho}(z,z^*)e^{\sum_{k=1}^\infty(-\alpha_k^{(1)*} z_k + \alpha_k^{(1)} z_k^*)} = f(z,z^*) = (\Phi, e^{\sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)}\Phi) \tag{54}$$

is a Fock density. Let $\alpha^{(2)} - \alpha^{(1)} \in l^2$ and

$$\Phi_1 = e^{\sum_k ((\alpha_k^{(1)} - \alpha_k^{(2)})\hat{a}_k^+ - ((\alpha_k^{(1)} - \alpha_k^{(2)}))^* \hat{a}_k^-)}\Phi.$$

One has

$$\tilde{\rho}(z,z^*)e^{\sum_{k=1}^\infty(-\alpha_k^{(2)*} z_k + \alpha_k^{(2)} z_k^*)} = (\Phi_1, e^{\sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)}\Phi_1). \tag{55}$$

Therefore, $\tilde{\rho}$ is an $\alpha^{(2)}$-density.

Let $\tilde{\rho}$ be an $\alpha^{(2)}$-density, $z \in l^2$ and

$$z_k^{(n)} = z_k, k \le n, \qquad z_k^{(n)} = 0, k > n.$$

It follows from eqs.(54) and (55) that

$$\frac{(\Phi_1, e^{\varepsilon \sum_k (z_k^{(n)} \hat{a}_k^+ - z_k^{(n)*} \hat{a}_k^-)}\Phi_1)}{(\Phi, e^{\varepsilon \sum_k (z_k^{(n)} \hat{a}_k^+ - z_k^{(n)*} \hat{a}_k^-)}\Phi)} = e^{\varepsilon \sum_k ((\alpha_k^{(2)} - \alpha_k^{(1)}) z_k^{(n)*} - (\alpha_k^{(2)*} - \alpha_k^{(1)*}) z_k^{(n)})}. \tag{56}$$

It follows from the corollary 3 of lemma 1 from the Appendix that

$$(\Phi_1, e^{\varepsilon \sum_k (z_k^{(n)} \hat{a}_k^+ - z_k^{(n)*} \hat{a}_k^-)}\Phi_1) \to_{n\to\infty} (\Phi_1, e^{\varepsilon \sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)}\Phi_1)$$
$$(\Phi, e^{\varepsilon \sum_k (z_k^{(n)} \hat{a}_k^+ - z_k^{(n)*} \hat{a}_k^-)}\Phi) \to_{n\to\infty} (\Phi, e^{\varepsilon \sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)}\Phi)$$

Corollary 3 also implies that for sufficiently small ε

$$(\Phi_1, e^{\varepsilon \sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)}\Phi_1) \ne 0,$$
$$(\Phi, e^{\varepsilon \sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)}\Phi) \ne 0.$$

This implies that

$$(\Phi_1, e^{\varepsilon \sum_k (z_k^{(n)} \hat{a}_k^+ - z_k^{(n)*} \hat{a}_k^-)}\Phi_1) \ne 0,$$
$$(\Phi, e^{\varepsilon \sum_k (z_k^{(n)} \hat{a}_k^+ - z_k^{(n)*} \hat{a}_k^-)}\Phi) \ne 0$$

at $n \ge n_1$. Therefore, the eft-hand side of eq.(56) tends to

$$\frac{(\Phi_1, e^{\varepsilon \sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)}\Phi_1)}{(\Phi, e^{\varepsilon \sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)}\Phi)}.$$

This means that the limit

$$\lim_{n\to\infty} \sum_k ((\alpha_k^{(2)} - \alpha_k^{(1)}) z_k^{(n)*} - (\alpha_k^{(2)*} - \alpha_k^{(1)*}) z_k^{(n)})$$

exists for all $z \in l^2$. Choose $z = i(\alpha^{(2)} - \alpha^{(1)})$. We see that $(\alpha^{(2)} - \alpha^{(1)} \in l^2$. Statement is proved.

We see that the notion of "density" (50) is useful in order to specify states corresponding to different representations of CCR. One can even investigate the case when the representation is time-dependent.

3.3.2 Evolution of density matrix

Let us write down the evolution equation for the average (50) for the system (36).

From eq.(37) one has

$$ir\dot{\tilde{h}}o =< [e^{\sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)}; H] > . \quad (57)$$

for the Fock density case. Eq. (57) can be postulates for the general case as well. It follows from CCR that

$$f(a^+, a^-) e^{\sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)} = e^{\sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)} f(a^+ + \alpha^*, a^- + \alpha).$$

This implies that

$$[H; e^{\sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)}] = e^{\sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)} \left[\Omega_k z_k^* \hat{a}_k^- + \Omega_k z_k \hat{a}_k^+ + \Omega_k z_k^* z_k + \mu_k \mathcal{Q} \frac{z_k^* + z_k}{\sqrt{2\Omega_k}} \right].$$

Furthermore, it follows from CCR that

$$\frac{\partial}{\partial z_i} < e^{\sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)} >=< e^{\sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)} (a_i^+ + \frac{1}{2} z_i^*) >;$$

$$-\frac{\partial}{\partial z_i^*} < e^{\sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)} >=< e^{\sum_k (z_k \hat{a}_k^+ - z_k^* \hat{a}_k^-)} (a_i^- + \frac{1}{2} z_i^*) >;$$

Substituting these relations to the commutator, we obtain the following evolution equation:

$$-i\dot{\tilde{\rho}} = \sum_k \left(-\Omega_k z_k^* \frac{\partial}{\partial z_k^*} + \Omega_k z_k \frac{\partial}{\partial z_k} + \mu_k \mathcal{Q} \frac{z_k + z_k^*}{\sqrt{2\Omega_k}} \right) \tilde{\rho}.$$

Substitution

$$\tilde{\rho}_t(z, z^*) = e^{\sum_{k=1}^{\infty} (\alpha_k^*(t) z_k - \alpha_k(t) z_k^*)} f(z, z^*)$$

gives us eq.(39) on the function $\alpha_k(t)$ and the following equation on f:

$$-i\dot{f} = \left(-\Omega_k z_k^* \frac{\partial}{\partial z_k^*} + \Omega_k z_k \frac{\partial}{\partial z_k} \right) f. \quad (58)$$

Statement 3. Let $\tilde{\rho}_0$ be an $\alpha(0)$-density. Then $\tilde{\rho}_t$ is an $\alpha(t)$-density, where $\alpha(t)$ is given by eq.(39).

Proof. It is sufficient to check that the property that f is a Fock density is invariant under evolution (58). Let

$$f_t = (\Phi_t, e^{\sum_{k=1}^{\infty} (\hat{a}_k^+ z_k - \hat{a}_k^- z_k^*)}, \Phi_t)$$

where $\Phi_t = e^{-\sum_k \Omega_k \hat{a}_k^+ \hat{a}_k^-} \Phi_0$. Note that f_t obeys eq.(58). Statement is proved.

3.3.3 Time evolution of the representation

According to statement 2, the function α_k specifying the choice of the representation is defined up to an element from l^2. We say that $\alpha \sim \alpha'$ if $\alpha - \alpha' \in l^2$. Denote the class of equivalence as $[\alpha]$.

Let the condition (23) be satisfied. In this case the quantity $\gamma_k^l(t)$ entering to eq.(42) is an element of l^2, so that

$$[\alpha_k(t)] = [(\alpha_k(0) - \beta_k^l(0)) e^{-i\Omega_k t} + \beta_k^l(t)],$$

where quantities $\beta^l_k(t)$ are expressed via $\mathcal{Q}(t)$, ..., $\mathcal{Q}^{(l-2)}(t)$. We see that in general case the representation of CCR at time moment t depends not only on the value of $\mathcal{Q}^{(s)}(t)$ but also on values $\mathcal{Q}^{(s)}(0)$. However, there is a *special* case when the representation depends only on the derivatives of $\mathcal{Q}$ at the same time moment. Such a case corresponds to the following choice of the initial representation:

$$\alpha_k(0) - \beta^l_k(0) \in l^2.$$

This implies that

$$\alpha_k(t) - \beta^l_k(t) \in l^2,$$

so that

$$[\alpha_k(t)] = [\beta^l_k(t)]. \tag{59}$$

If the condition (23) is satisfied at $l = 2$, the formula (59) for the representation takes the form:

$$[\alpha_k(t)] = -\frac{\mu_k}{\sqrt{2\Omega_k^3}}\mathcal{Q}(t). \tag{60}$$

This choice of the representation is in agreement with papers [4, 5]. In these articles the processes in strong nonstationary electromagnetic and gravitational fields were considered. It was suggested to consider the representation obtained by the diagonalization procedure of the Hamiltonian at each time moment.

We see also that if the condition (23) is not satisfied at $l = 2$, the prescription (60) of [4, 5] is not valid.

4 Explanation of the paradox

We have constructed in section 2 the renormalized Hilbert state space and the renormalized evolution operator for the model (1) which is unitary. This implies that it is sufficient to use one representation of CCR. This is in agreement with the Whightman axiomatic approach. However, section 3 tells us that the model (1) in the strong external field in the leading order in g is unusual: one should choose different representations of CCR at different time moments. Let us discuss this paradox.

4.1 Extraction of the c-number from the field in the cutoffed theory

The procedure of extracting the c-number component of the field Q is justified within the Hamiltonian framework as follows [37, 38, 39]. Consider the regularized theory with fixed Λ. At the fixed moment of time the state of the system is specified by sets $\mathcal{P} = (\mathcal{P}_0, ..., \mathcal{P}_{l-1})$ and $\mathcal{Q} = (\mathcal{Q}, ..., \mathcal{Q}^{(l-1)})$ and regular as $g \to 0$ vector X of the Fock space which can be identified with the state in the external field $\mathcal{Q}$. The solution of the regularized Schrodinger equation (5) depends on the small parameter g as

$$\Psi_t = e^{\frac{i}{g^2}S_t} U_g[\mathcal{P}_t, \mathcal{Q}_t] X_t \tag{61}$$

where the index Λ is omitted,

$$U_g[\mathcal{P}, \mathcal{Q}] = \exp\frac{i}{g}\left(\sum_{s=0}^{l-1}(\mathcal{P}_s\hat{V}_s - \mathcal{Q}^{(s)}\hat{P}_s)\right).$$

Substituting vector (61) to eq.(5), making use of the relations

$$(U_g[\mathcal{P},\mathcal{Q}])^+\hat{Q}^{(s)}U_g[\mathcal{P},\mathcal{Q}] = \hat{Q}^{(s)} + \frac{1}{g}\mathcal{Q}^{(s)},$$
$$(U_g[\mathcal{P},\mathcal{Q}])^+\hat{P}_sU_g[\mathcal{P},\mathcal{Q}] = \hat{P}_s + \frac{1}{g}\mathcal{P}_s, \tag{62}$$

we obtain that the number S_t is the action on the classical trajectory satisfying eqs.(19) as $g=0$:

$$\dot{V}_j = V_{j+1}, j=\overline{0,l-1}; \dot{V}_{l-1} = (-1)^{l-1}\frac{P_{l-1}}{z_l};$$
$$-\dot{P}_j = (-1)^j(z_j+\delta z_j)V_j + P_{j-1}, j=\overline{1,l-1}; \tag{63}$$
$$-\dot{P}_0 = (z_0+\delta z_0)Q.$$

At $g\to 0$ the vector X_t obeys the equation:

$$i\dot{X}_t = [H_Q + H_q + \sum_{k=1}^{\infty}\mu_k Q q_k + E_0]X_t.$$

for a c-number E_0. The operator entering to the right-hand side of this equation is presented as a sum of the term corresponding to the field Q and Hamiltonian (36).

Note that the classical solution $\mathcal{Q}(t)$ can be expressed as a linear combination of the initial conditions for the system (63):

$$\mathcal{Q}(t) = \sum_{s=0}^{l-1}(a_s(t)\mathcal{Q}_s + b_s(t)\mathcal{P}_s) \tag{64}$$

Analogously, the Heisenberg operators $\hat{V}_s(t) = e^{iH_Qt}\hat{V}_se^{-iH_Qt}$ and $\hat{P}_s(t) = e^{iH_Qt}\hat{P}_se^{-iH_Qt}$ obey the analog of the system (63) for operator functions. Therefore, for $\hat{V}_0 = \hat{Q}$ we have

$$\hat{Q}(t) = \sum_{s=0}^{l-1}(a_s(t)\hat{V}_s + b_s(t)\hat{P}_s). \tag{65}$$

Eqs. (64) and (65) imply that

$$U_g^+[\mathcal{P},\mathcal{Q}]g\hat{Q}(t)U_g[\mathcal{P},\mathcal{Q}] = \mathcal{Q}(t) + O(g), \tag{66}$$

The property (66) will be used in the next subsection.

4.2 The $\Lambda\to\infty$-limit

Consider the limit $\Lambda\to\infty$. According to section 2, the vector Ψ^t_Λ should depend on Λ according to (6), while Φ^t_Λ should be regular as $\Lambda\to\infty$. Without loss of generality, consider the fixed moment of time t; index t will be omitted. Construct the state in the external field obeying the condition (6):

$$T_\Lambda e^{\frac{i}{g^2}S}U_g[\mathcal{P},\mathcal{Q}]Y_\Lambda. \tag{67}$$

The vector Y_Λ is regular as $\Lambda\to\infty$, $Y_\Lambda \to_{\Lambda\to\infty} Y$. Show that for finite values of Λ, the vector (67) is of the type (61) and therefore corresponds to the external field. Comparing eqs. (67) and (61), one obtains:

$$X_\Lambda = T_\Lambda(\mathcal{P},\mathcal{Q})Y_\Lambda,$$

where

$$T_\Lambda(\mathcal{P},\mathcal{Q}) = U_g^+[\mathcal{P},\mathcal{Q}]T_\Lambda U_g[\mathcal{P},\mathcal{Q}]. \tag{68}$$

The vector Y can be viewed as a renormalized state in the external field. Let us check that the operator $T_\Lambda(\mathcal{P},\mathcal{Q})$ is regular as $\Lambda = const$, $g\to 0$.

It follows from eqs.(10) and (11) that it transforms the initial condition for the equation

$$i\frac{d\tilde{\Phi}^t_\Lambda}{dt} = U_g^+[\mathcal{P},\mathcal{Q}]e^{i\hat{H}_0 t}(g\xi_-(t)\hat{H}_1^\Lambda + \hat{H}^\Lambda_{ct}[t,g(\cdot)])e^{-i\hat{H}_0 t}U_g[\mathcal{P},\mathcal{Q}]\tilde{\Phi}^t_\Lambda \tag{69}$$

at $t=-\infty$ to the solution of this equation as $t=0$,

$$\tilde{\Phi}^0_\Lambda = T_\Lambda(\mathcal{P},\mathcal{Q})\tilde{\Phi}^{-\infty}_\Lambda.$$

Making use of eq.(17), take eq.(69) to the form

$$\begin{aligned} i\tfrac{d\tilde{\Phi}^t_\Lambda}{dt} &= U_g^+[\mathcal{P},\mathcal{Q}](g\xi_-(t)\textstyle\sum_k \mu_k \hat{Q}(t)\frac{\hat{a}_k^+e^{i\Omega_k t}+\hat{a}_k^- e^{-i\Omega_k t}}{\sqrt{2\Omega_k}} \\ &+ \textstyle\sum_{s=0}^{l-1}\frac{(-1)^s}{2}\delta z_s(\hat{Q}^{(s)}(t))^2)U_g[\mathcal{P},\mathcal{Q}]\tilde{\Phi}^t_\Lambda \end{aligned} \tag{70}$$

In the leading order in g one has

$$i\frac{d\tilde{\Phi}^t_\Lambda}{dt} = (\xi_-(t)\sum_k \mu_k \mathcal{Q}(t)\frac{\hat{a}_k^+e^{i\Omega_k t}+\hat{a}_k^- e^{-i\Omega_k t}}{\sqrt{2\Omega_k}} + \sum_{s=0}^{l-1}\frac{(-1)^s}{2g^2}\delta z_s(\mathcal{Q}^{(s)}(t))^2)\tilde{\Phi}^t_\Lambda$$

The evolution operator $V_{t,-\infty}$ for this equation which transforms the initial state at $t=-\infty$ to the solution has the form

$$V_{t,-\infty} = c_t \exp[\sum_k(\alpha_k(t)\hat{a}_k^+ - \alpha_k^*(t)\hat{a}_k^-)] \tag{71}$$

where

$$\alpha_k^\Lambda(t) = -i\int_{-\infty}^t d\tau \xi_-(\tau)\mathcal{Q}(\tau)\frac{\mu_k^\Lambda}{\sqrt{2\Omega_k}}e^{i\Omega_k\tau} \tag{72}$$

$$c_t = \exp(\int_{-\infty}^t d\tau[-\frac{i}{g^2}\sum_{s=0}^{l-1}\frac{(-1)^s}{2}\delta z_s(\mathcal{Q}^{(s)}(\tau))^2 + \frac{1}{2}(\alpha_k^*(\tau)\dot{\alpha}_k(\tau) - \dot{\alpha}_k^*(\tau)\alpha_k(\tau))])$$

In the leading order in g $T_\Lambda(\mathcal{P},\mathcal{Q}) = V_{0,-\infty}$. Note that at $\Lambda\to\infty$ the operator (71) may be singular, since the series $\sum_k|\lim_{\Lambda\to\infty}\alpha_k^\Lambda|^2$ may diverge.

4.3 Representations of CCR in the external field

The average (50) for the semiclassical state (67) has the form:

$$< e^{\sum_k(z_k\hat{a}_k^+ - z_k^*\hat{a}_k^-)} >= (Y, e^{\sum_k(z_k\pi_{\mathcal{P},\mathcal{Q}}(a_k^+) - z_k^*\pi_{\mathcal{P},\mathcal{Q}}(a_k^-))}Y). \tag{73}$$

where

$$\pi_{\mathcal{P},\mathcal{Q}}(a_k^\pm) = (T_\Lambda(\mathcal{P},\mathcal{Q}))^+\hat{a}_k^\pm(T_\Lambda(\mathcal{P},\mathcal{Q})). \tag{74}$$

It follows from eq.(71) that

$$\pi_{\mathcal{P},\mathcal{Q}}(a_k^+) = \hat{a}_k^+ + \alpha_k^*, \qquad \pi_{\mathcal{P},\mathcal{Q}}(a_k^-) = \hat{a}_k^- + \alpha_k$$

for α_k of the form (72) as $\Lambda=\infty$. We see that the function (73) is an α-density. Integrating eq. (72) by parts, we obtain that

$$[\alpha_k] = [-\frac{\mu_k}{\sqrt{2\Omega_k}}\sum_{s=0}^{l-2} i^s\mathcal{Q}^{(s)}(0)\frac{1}{\Omega_k^{s+1}}],$$

provided that the condition (23) is satisfied.

This representation coincides with eq.(59). We see from the direct analysis of the exact quantum model that one should choose only such initial condition for the representation that relation (59) is satisfied.

It is interesting to note that all the representations $\pi_{\mathcal{P},\mathcal{Q}}$ are *equivalent*. Namely,

$$\pi_{\mathcal{P},\mathcal{Q}}(a_k^\pm) = U_g^+[\mathcal{P},\mathcal{Q}]\pi_0(a_k^\pm)U_g[\mathcal{P},\mathcal{Q}] \tag{75}$$

However, the operator U_g is singular in g and does not possess a limit $g \to 0$. This means that representations of CCR appears to be equivalent in the exact theory and not equivalent in the approximate theory.

This fact can be understood as follows. According to subsection 2.3, the structure of $\pi_0(a_k^\pm)$ is

$$\pi_0(a_k^-) = \hat{a}_k^- + \sum_{m=1}^{\infty}(A_{km}\hat{a}_m^- + B_{km}\hat{a}_m^+) + \sum_{s=0}^{l-1}(C_{ks}\hat{V}_s + D_{ks}\hat{P}_s)$$

for some coefficients A_{km}, B_{km}, C_{ks}, D_{ks}. It follows from section 2 that at fixed k the vectors with components A_{km}, B_{km}, C_{ks}, D_{ks} are regular as $\Lambda \to \infty$. Consider the limit $\Lambda \to \infty$. Eqs.(18) imply that the coefficients C_{km} and D_{km} are of order g, while A_{kl} and B_{kl} are of the order g^2. It follows from eq.(62) that the representations (75) have the following structure:

$$\pi_{\mathcal{P},\mathcal{Q}}(a_k^-) = \hat{a}_k^- + \beta_k + O(g) \tag{76}$$

where $\beta_k = g^{-1}\sum_{s=0}^{l-1}(C_{ks}\mathcal{Q}^{(s)} + D_{ks}\mathcal{P}_s) + O(g)$ Eqs.(71) and (74) imply that β_k is

$$\beta_k = \alpha_k(0).$$

Eq. (76) is in agreement with obtained in section 3.

Thus, the representations (75) viewed in the leading order in g are nonequivalent if $\sum_k |\beta_k|^2 = \infty$ (i.e. if the condition (43) is not satisfied). However, in the exact theory they are equivalent and related with the help of the unitary operator which is singular in g.

One can expect that analogous difficulties corresponding to extracting the classical component of the field arise in QED.

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Appendix

In this appendix the auxiliary mathematical statements are presented.

Lemma 1. *Let U_n be a sequence of unitary operators in the Fock space, and the sequence of their Wich symbols $U_n(z^*,w)$ converges to 1 as $n \to \infty$ at arbitrary $z,w \in l^2$. Then $U_n \to 1$ in strong sense.*

Proof.

It follows from the Banach-Shteingaus (?) theorem (see, for example, [40]) that it is sufficient to check that at some dense subset of the Fock space

$$||U_n f - f|| \to_{n\to\infty} 0. \tag{77}$$

This subset is chosen as a set of all finite linear combinations of the coherent states

$$f = \sum_{i=1}^{K}\alpha_i \exp(\sum_{m=1}^{\infty} z_i^m B_m^+)|0> . \tag{78}$$

It follows from the unitarity of the operator U_n that for vector (78) the property (77) takes the form

$$\sum_{ij=1}^{K} \alpha_i^* \alpha_j \exp(\sum_{m=1}^{\infty} z_i^{m*} z_j^m)(2 - U_n^*(z_i^*, z_j) - U(z_i^*, z_j)) \rightarrow_{n\rightarrow\infty} 0,$$

It is satisfied since the Wick symbol converges.

Corollary 1. *Let U_n be a sequence of unitary operators corresponding to linear canonical transformations*

$$U_n^{-1} B_m^+ U_n = \sum_{k=1}^{\infty} ((G_n)_{mk} B_k^+ + (F_n^*)_{mk} B_k^-), \tag{79}$$

the operators F_n and G_n with matrices $(F_n)_{mk}$ and $(G_n)_{mk}$ converge as $n \rightarrow \infty$ in the operator norm $F_n \rightarrow_{n\rightarrow\infty} 0$, $G_n \rightarrow_{n\rightarrow\infty} 1$; for some vectors of the Fock space f and g $(f, U_n g) \rightarrow (f, g)$. Then $U_n \rightarrow 1$ in a strong sense.

Proof. According to [8], the Wick symbol of the operator U_n has the form (24) up to a multiplier. It follows from lemma 1 that $\frac{U_n}{<0|U_n|0>} \rightarrow 1$. Therefore, $\frac{(f,U_n g)}{<0|U_n|0>} \rightarrow \frac{(f,g)}{<0|U_n|0>}$, $< 0|U_n|0 >\rightarrow 1$ and $U_n \rightarrow 1$.

Corollary 2. *Let U_n be a sequence of unitary operators corresponding to linear canonical transformations (79), the unitary operator U corresponds to the canonical transformation*

$$U^{-1} B_m^+ U = \sum_{k=1}^{\infty} (G_{mk} B_k^+ + F_{mk}^* B_k^-),$$
$$F_n \rightarrow_{n\rightarrow\infty} F, G_n \rightarrow_{n\rightarrow\infty} G; < 0|U_n|0 >\rightarrow < 0|U|0 > .$$

Then $U_n \rightarrow 1$ in a strong sense.

Corollary 3. *Let U_n be a sequence of unitary operators*

$$U_n = \exp(\sum_{k=1}^{\infty} (z_k^{(n)} \hat{a}_k^+ - z_k^{(n)*} \hat{a}_k^-)),$$

while

$$U = \exp(\sum_{k=1}^{\infty} (z_k \hat{a}_k^+ - z_k \hat{a}_k^-)),$$

where $z^{(n)} \in l^2$, $z \in l^2$, $||z^{(n)} - z|| \rightarrow_{n\rightarrow\infty} 0$. Then $U_n \rightarrow 1$ in a strong sense.

To prove the corollaries, it is sufficient to consider the sequence of unitary operators $U_n U^{-1}$ and use the statement of corollary 1.

Lemma 2. *Let A_n and B_n be Hilbert-Schmidt operators, which converge in the norm (26) to operators A and B, while the operator $1+AB$ is invertible. Then $det(1+A_n B_n) \rightarrow_{n\rightarrow\infty} det(1 + AB)$.*

Proof. The property (13) $Tr|XYZ| \leq ||X||_2||Y||||Z||_2$ imply that for the quantity $c_n = \frac{det(1+A_n B_n)}{det(1+AB)}$

$$|lnc_n| = | \int_0^1 d\tau Tr([1 + \tau(1 + AB)^{-1}(A_n B_n - AB)]^{-1}$$
$$(1 + AB)^{-1}(A_n B_n - AB))| \leq$$
$$max_{\tau\in(0,1)}||[1 + \tau(1 + AB)^{-1}(A_n B_n - AB)]^{-1}||||(1 + AB)^{-1}||$$
$$(||A_n - A||_2||B_n||_2 + ||A||_2||B_n - B||_2) \rightarrow_{n\rightarrow\infty} 0.$$

Lemma is proved.

Consider the sequence of the Volterra integral equations

$$q_n(t) = \alpha_n(t) + \int_{-T}^{t} d\tau f_n(t, \tau) q_n(\tau),$$

for q_n at $[-T, T]$. Denote by $G_n(t, \tau)$ the Green functions for these equations which are defined form the relation

$$q_n(t) = \int_{-T}^{t} G_n(t, \tau)\alpha_n(\tau)d\tau$$

Lemma 3. *Let the sequences of functions f_n and α_n uniformly converge to f and α. Then the sequence $G_n(t, \tau)$ uniformly converge to the Green function of the equation*

$$q(t) = \alpha(t) + \int_{-T}^{t} d\tau f(t, \tau)q(\tau),$$

and the sequence q_n uniformly converge to q.

To prove the lemma, it is sufficient to use the explicit form of the Green function which is obtained from the iteration procedure of [36].

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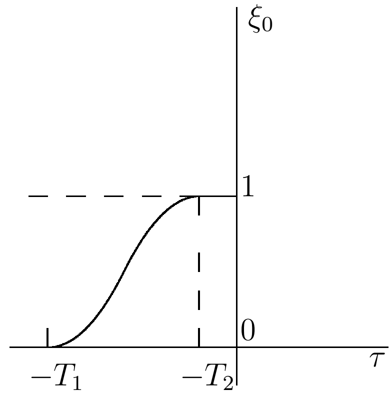

Fig.1